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Weakly Wandering Vectors for Compactly Generated Groups of Unitary Operators

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1. INTRODUCTION

Let $\mathbf{H}$ be a complex Hilbert space and G an Abelian, locally compact group. Further, let $\{T_g \mid g \in G\}$ be a group of unitary operators on $\mathbf{H}$ indexed by G, such that the map $g \to T_g$ is weakly continuous. Denote by $\mathbf{H}_c^T$ the continuous part of $\mathbf{H}$ (with respect to T); that is, the orthogonal complement of the space of all eigenvectors of $\{T_g \mid g \in G\}$.

We consider the relation of $\mathbf{H}_c^T$ to certain elements of $\mathbf{H}$ called weakly wandering vectors. We call $f \in \mathbf{H}$ weakly wandering if there exists a sequence (g_n) of group elements $g_n \to \infty$ such that

$$T_{g_m} f \perp T_{g_n} f \qquad \text{for } m \neq n.$$

In this paper we prove that, for G compactly generated, the set of weakly wandering vectors is dense in $\mathbf{H}_c^T$.

This theorem generalizes a recent result of *U*. Krengel [6, Theorem 1.1]. Krengel's result deals with the case where the group is the "discrete" group $\{U^k \mid k \in Z\}$ (the iterates of a single unitary operator). Among other concepts, our proof relies on the notions of invariant mean of almost periodic functions and the Fourier transform of the spectral measure.

2. GROUPS OF OPERATORS: SPECTRAL PROPERTIES

Suppose we are given an Abelian, locally compact group G and a homomorphism $T: g \to T_g$ of G into $\mathscr{U}(\mathbf{H})$, where $\mathscr{U}(\mathbf{H})$ is the group of unitary operators on $\mathbf{H}$. The set $\{T_g \mid g \in G\}$ forms a subgroup, called a group of unitary operators indexed by G. Where there is no ambiguity, we will denote the group $\{T_g \mid g \in G\}$ simply as T.

$\{T_g \mid g \in G\}$ is called a continuous group of operators if the map $T: g \to T_g$ is continuous in the weak topology of $\mathscr{U}(\mathbf{H})$; that is, $g \to \langle T_g f, h\rangle$ is continuous for each $f, h \in \mathbf{H}$. It is easily shown that the map $(g, f) \to T_g f$ of $G \times \mathbf{H} | \to \mathbf{H}$ is then continuous.

We first need a few concepts from harmonic analysis. We define a character of a group G as a continuous homomorphism from G into the complex numbers with absolute value 1. The set of characters of G, denoted by G^*, is called the dual group. G^* forms a group under multiplication.

If G is Abelian, locally compact, then G^* also has these properties. In fact, in this case the Pontryagin–van Kampen Duality Theorem states that G and G^{**} are topologically isomorphic [3].

Denote by μ a Haar measure on G. For $f \in L^1(d\mu)$ we define the Fourier transform $\hat{f} : \chi \to \hat{f}(\chi)$ from G^* into C, the complex numbers, as:

$$\hat{f}(\chi) = \int_G \overline{\chi(g)} f(g) \, d\mu(g) \qquad \text{for } \chi \in G^*.$$

Similarly, for λ a bounded, complex-valued measure on G, the transform $\hat{\lambda}: \chi \to \hat{\lambda}(\chi)$ from G^* into C is defined as:

$$\hat{\lambda}(\chi) = \int_G \overline{\chi(g)} \, d\lambda(g) \qquad \text{for } \chi \in G^*.$$

Returning to the group $\{T_g \mid g \in G\}$ of unitary operators, we make the following definitions:

DEFINITION 1. A vector $f \in \mathbf{H}$, $f \neq 0$ is called an eigenvector of T if, for some $\chi \in G^*$,

$$T_g f = \chi(g) f \qquad \text{for all } g \in G.$$

χ is called the eigenvalue associated with f.

Equivalently, we could define an eigenvector of $\{T_g \mid g \in G\}$ as any $f \in \mathbf{H}$, $f \neq 0$ which is an eigenvector of the operator T_g for each $g \in G$. For then we have

$$T_g f = c_g f \qquad \text{for } c_g \text{ complex}, \qquad | c_g | = 1.$$

Group properties and continuity of $g \to T_g f$ then show that $g \to c_g$ is a character of G.

For notation, let $\mathbf{H}_e{}^T$ be the space spanned by the eigenvectors of $\{T_g \mid g \in G\}$, $G(\chi)$ the space spanned by the eigenvectors with eigenvalue χ.

We wish to study properties of $\langle T_g f, h \rangle$. A useful tool is the associated spectral measure. Let $\mathscr{B}^*$ be the Borel sets of G^*, $\mathscr{P}$ the group of orthogonal projections on $\mathbf{H}$. We then define [2, p. 58–62].

DEFINITION 2. A regular spectral measure in G^* is a map $E: B \to E(B)$ from $\mathscr{B}^*$ into $\mathscr{P}$ such that:

(a) $E(G^*) = I$

(b) $E(B) = \sum_i E(B_i)$ for $B = \bigcup_{i=1}^{\infty} B_i$, B_i disjoint

$$\Big(E(B)f = \sum_i E(B_i)f \text{ for all } f \in \mathbf{H}\Big)$$

(c) $E(B_0) = \bigvee_{\substack{B \text{compact} \\ B \subseteq B_0}} E(B).$

From the definition, it follows that

$$E(B_1)\, E(B_2) = E(B_1 \cap B_2) \qquad \text{for } B_1, B_2 \in \mathscr{B}^*$$
$$E(\varnothing) = 0.$$

With each group of unitary operators $\{T_g \mid g \in G\}$, there is associated a unique regular spectral measure, as proved in the following Spectral Theorem [1]:

PROPOSITION 1. *Let G be an Abelian, locally compact group and $\{T_g \mid g \in G\}$ a continuous group of unitary operators on $\mathbf{H}$. There exists a unique regular spectral measure E: $B \to E(B)$ such that*

$$T_g = \int_{G^*} \chi(g)\, dE(\chi) \qquad \textit{for all } g \in G$$

in the sense that, for all f, $h \in \mathbf{H}$,

$$\langle T_g f, h\rangle = \int_{G^*} \chi(g)\, d\langle E(\chi)f, h\rangle \qquad \textit{for all } g \in G.$$

An operator commutes with every T_g if and only if it commutes with every $E(B)$, $B \in \mathscr{B}^*$.

Remark 1. $E(\chi_1)$ is the projection onto $G(\chi_1)$ for $\chi_1 \in G^*$. For, given f, $h \in \mathbf{H}$,

$$\langle T_g E(\chi_1)f, h\rangle = \int_{G^*} \chi(g)\, d\langle E(\chi)\, E(\chi_1)f, h\rangle$$
$$= \chi_1(g)\, \langle E(\chi_1)f, h\rangle.$$

Therefore,

$$T_g E(\chi_1)f = \chi_1(g)\, E(\chi_1)f \qquad \text{and} \qquad E(\chi_1)f \in G(\chi_1).$$

Conversely, let P denote the projection onto $G(\chi_1)$. P commutes with T_g for all g, and therefore commutes with each $E(\chi)$. For $f, h \in \mathbf{H}$,

$$\langle \chi_1(g)\, Pf, h\rangle = \langle T_g Pf, h\rangle = \int_{G^*} \chi(g)\, d\langle E(\chi)\, Pf, h\rangle$$

$$= \int_{G^*} \chi(g)\, d\langle PE(\chi) f, h\rangle = \chi_1(g)\, \langle E(\chi_1) f, h\rangle.$$

Therefore,

$$Pf = E(\chi_1) f \qquad \text{for } f \in \mathbf{H}.$$

3. Invariant Means and T

In the following we consider $\mathscr{B}$, the Banach space of bounded, measurable functions on an Abelian, topological group G, with sup norm $\|\ \|_\infty$. We wish to define an invariant mean on some subspace of $\mathscr{B}$ (for a detailed discussion of invariant means, see [3, pp. 230–262]). For this, we define a special type of sequence in G.

Definition 3. A sequence $\mathscr{C} = (C_n)$ of open subsets of G is called rapidly expanding if:

(1) C_n has compact closure for all n

(2) $C_1 \subseteq C_2 \subseteq \cdots \subseteq \bigcup_n C_n = G$

(3) $\lim_{n\to\infty} \dfrac{\mu[(g + C_n) \cap C_n']}{\mu(C_n)} = 0$ for all $g \in G$.

If G is countable at infinity, such a $\mathscr{C}$ can always be constructed. Assume $\mathscr{C} = (C_n)$ is a rapidly expanding sequence in G. Define

$$M_{\mathscr{C}}(f) = \limsup_{n\to\infty} \frac{1}{\mu(C_n)} \int_{C_n} f(g)\, d\mu(g)$$

$M_{\mathscr{C}}$ is then an invariant mean on $\mathscr{B}$ [3].

Indeed, Hewitt and Stromberg show that [4]:

Lemma 1. *Let G be an Abelian, locally compact group countable at infinity. Let λ be a complex-valued, regular Borel measure on G^* and $\mathscr{C}$ a rapidly expanding sequence on G. Then the Fourier transform $g \to \hat{\lambda}(g)$*

$$\hat{\lambda}(g) = \int_{G^*} \overline{\chi(g)}\, d\lambda(\chi)$$

is such that

(1) $M_{\mathscr{C}}(\hat{\lambda}) = \lambda(e)$, *e the identity of G*

(2) $M_{\mathscr{C}}(|\hat{\lambda}|^2) = \sum_{\chi \in G^*} |\lambda(\chi)|^2$.

(3) λ *is continuous* $\Leftrightarrow M_{\mathscr{C}}(|\hat{\lambda}|) = 0$

$$\Leftrightarrow M_{\mathscr{C}}(|\hat{\lambda}|^2) = 0.$$

Let us now define a special measure

$$\lambda(A) = \langle \overline{E(A)f, h} \rangle \qquad A \subseteq \mathscr{B}^*.$$

λ is then a regular Borel measure, and we can apply Lemma 1. In addition, we find

$$\hat{\lambda}(g) = \langle \overline{T_g f, h} \rangle.$$

Define $\mathbf{H}_c^T = (\mathbf{H}_e^T)^\perp$, the orthogonal complement of $\mathbf{H}_e^T$ in $\mathbf{H}$. $\mathbf{H}_c^T$ is called the continuous part of $\mathbf{H}$ with respect to T.

We have the following.

Proposition 2. *Let G be an Abelian, locally compact group countable at infinity, and correspondingly let* $\{T_g \mid g \in G\}$ *be a continuous group of unitary operators. Then for an element* $f \in \mathbf{H}$, *the following are equivalent.*

(1) $f \in \mathbf{H}_c^T$.

(2) $E(\chi)f = 0$ *for all* $\chi \in G^*$.

(3) $M_{\mathscr{C}}(|\langle T_g f, f\rangle|) = 0$.

(4) $M_{\mathscr{C}}(|\langle T_g f, h\rangle|) = 0$ *for all* $h \in \mathbf{H}$.

Proof. Using Lemma 1, we have:

$$\begin{aligned} M_{\mathscr{C}}(|\langle T_g f, h\rangle|) &= 0 \qquad \text{for all } h \in \mathbf{H} \Rightarrow, \\ M_{\mathscr{C}}(|\langle T_g f, f\rangle|) &= 0 \Rightarrow, \\ \langle E(\chi)f, f\rangle = \langle E(\chi)f, E(\chi)f\rangle &= 0 \qquad \text{for all } \chi \in G^*, \\ \Rightarrow E(\chi)f = 0 \Rightarrow \langle E(\chi)f, h\rangle &= 0 \qquad \text{for all } h \in \mathbf{H}, \\ \Rightarrow M_{\mathscr{C}}(|\langle T_g f, h\rangle|) &= 0. \end{aligned}$$

Since $E(\chi)$ is the projection onto $G(\chi)$ [see Remark 1],

$$f \in \mathbf{H}_c^T \Leftrightarrow E(\chi)f = 0 \qquad \text{for all } \chi \in G^*.$$

Remark 2. By Proposition 2, we find that for G *compact* we have $\mathbf{H}_c^T = \{0\}$. For, since $\mu(G) < \infty$, we may define $C_n = G$, giving

$$M_{\mathscr{C}}(|\langle T_g f, f\rangle|) = \frac{1}{\mu(G)} \int_G |\langle T_g f, f\rangle| \, d\mu(g).$$

Therefore we have $M_{\mathscr{C}}(|\langle T_g f, f\rangle|) = 0$ if and only if $f = 0$.

In order to make further use of Proposition 2, we explore the effects of hypothesis 3 on the function $g \to |\langle T_g f, f\rangle|$. We will show that it decreases as g gets "further away" except on a "small" set.

In the following, let G and $\mathscr{C}$ be such that

G is Abelian, locally compact,
noncompact, countable at infinity with Haar measure μ. $\quad (*)$
$\mathscr{C}$ is rapidly expanding.

We assume all the following sets to be measurable subsets of G. We define the *density* of a set in the following.

Definition 4. Given a measurable set $B \subseteq G$, we say that B has density α (with respect to $\mathscr{C}$) and we write $d(B) = \alpha$, if

$$\frac{\mu(B \cap C_n)}{\mu(C_n)} \to \alpha \qquad \text{as} \qquad n \to \infty.$$

The following properties hold.

(i) $d(B) = 0$ if $\mu(B) < \infty$.

(ii) $d(B') = 1$ if $d(B) = 0$.

(iii) $d\left(\bigcup_{i=1}^{n} B_i\right) = 0$ if $d(B_i) = 0$, $i = 1, \ldots, n$.

Property (iii) does not hold for infinite unions. However, we have the following useful lemma.

Lemma 2. *Let G and $\mathscr{C}$ satisfy $(*)$. Let (E_n) be a sequence of density zero sets. There exists a set B of density zero such that, for each j,*

$$\left(\bigcup_{i=1}^{j} E_i\right) \cap C'_{n_j} \subseteq B \qquad \textit{for some } C_{n_j} \in \mathscr{C}.$$

Proof. Since $d(\bigcup_{i=1}^{n} E_i) = 0$, we may assume $E_1 \subseteq E_2 \subseteq \cdots$, $d(E_n) = 0$. Define a sequence (N_j) such that $N_1 \leqslant N_2 \leqslant \cdots$ and

$$\mu(C_m \cap E_n) < \frac{1}{2^n} \mu(C_m), \qquad m \geqslant N_n. \tag{2.1}$$

Define

$$B_n = E_n \cap C'_{N_n} \qquad \text{and} \qquad B = \bigcup_{n=1}^{\infty} B_n .$$

We have for $N_{i-1} < N \leqslant N_i$, that

$$B \cap C_N = (E_1 \cap C'_{N_1} \cap C_N) \cup \cdots \cup (E_{i-1} \cap C'_{N_{i-1}} \cap C_N).$$

Since

$$(E_1 \cap C'_{N_2} \cap C_N) \subseteq E_2 \cap C'_{N_2} \cap C_N \subseteq \cdots,$$

we write

$$\begin{aligned} B \cap C_N &= (E_1 \cap C'_{N_1} \cap C_{N_2}) + \cdots + (E_{i-1} \cap C'_{N_{i-1}} \cap C_N) \\ &= (B_1 \cap C_{N_2}) + (B_2 \cap C_{N_3}) + \cdots + (B_{i-1} \cap C_N). \end{aligned}$$

By (2.1) we have

$$\frac{\mu(B \cap C_N)}{\mu(C_N)} \leqslant \frac{1}{2} \frac{\mu(C_{N_2})}{\mu(C_N)} + \frac{1}{2^2} \frac{\mu(C_{N_3})}{\mu(C_N)} + \cdots + \frac{1}{2^{i-1}} \frac{\mu(C_N)}{\mu(C_N)}.$$

Choosing j so $1/2^j < \epsilon/2$, N so $\mu(C_{N_j})/\mu(C_N) < \epsilon/2$,

$$\frac{\mu(B \cap C_N)}{\mu(C_N)} \leqslant \frac{\epsilon}{2} \left(\frac{1}{2} + \cdots + \frac{1}{2^j} \right) + \frac{1}{2^j} \left(\frac{1}{2} + \cdots \right) < \epsilon.$$

PROPOSITION 3. *Let G and $\mathscr{C}$ satisfy $(*)$. Let $f \in \mathscr{B}$, the bounded measurable functions on G. Then $M_{\mathscr{C}}(|f|) = 0$ if and only if there exists $B \subseteq G$ of density zero such that for any $\delta > 0$ there is $N_\delta > 0$ with*

$$|f(g)| < \delta \qquad \textit{for } g \notin (C_{N_\delta} \cup B).$$

Proof. Define $E_n = \{g \mid |f(g)| \geqslant 1/2^n\}$. For fixed n, we find

$$\begin{aligned} \frac{1}{2^n} \frac{\mu(C_m \cap E_n)}{\mu(C_m)} &\leqslant \frac{1}{\mu(C_m)} \int_{C_m \cap E_n} |f(g)| \, d\mu(g) \\ &\leqslant \frac{1}{\mu(C_m)} \int_{C_m} |f(g)| \, d\mu(g) \to 0 \qquad \text{as} \qquad m \to \infty. \end{aligned}$$

Therefore, $d(E_n) = 0$. By Lemma 2, there exists B, $d(B) = 0$, such that

$$E_n \cap C'_{N_n} \subseteq B \qquad \text{for } N_1 \leqslant N_2 \leqslant \cdots .$$

Given $\delta > 1/2^n$,

$$|f(g)| > \delta \Rightarrow g \in E_n \Rightarrow g \in C_{N_n} \cup B.$$

Therefore

$$g \notin (C_{N_n} \cup B) \Rightarrow |f(g)| < \delta.$$

Conversely, given such a B, find N so that

$$|f(g)| < \delta \qquad \text{for any } g \notin (C_N \cup B).$$

For $n \geqslant N$,

$$\begin{aligned}
&\int_{C_n} |f(g)|\, d\mu(g) \\
&\qquad \leqslant \left[\int_{C_N} |f(g)|\, d\mu(g) + \int_{B' \cap C_N' \cap C_n} |f(g)|\, d\mu(g) + \int_{C_n \cap B} |f(g)|\, d\mu(g)\right] \\
&\qquad \leqslant \|f\|_\infty \mu(C_N) + \delta\mu(C_n) + \|f\|_\infty \mu(C_n \cap B).
\end{aligned}$$

Therefore, we have

$$\limsup_{n\to\infty} \frac{1}{\mu(C_n)} \int_{C_n} |f(g)|\, d\mu(g) \leqslant \delta \qquad \text{as} \qquad n \to \infty.$$

Since δ was arbitrary, we have

$$M_{\mathscr{C}}(|f|) = 0.$$

Our proof is now complete.

Under further restrictions on G and $\mathscr{C}$, we may say more about the behavior of such functions.

Define

$$kD = \{kd \mid d \in D\} \qquad \text{for } D \subseteq G, \quad k \in Z_+ .$$

Suppose G and $\mathscr{C}$ satisfy $(*)$. We say that G and $\mathscr{C}$ also satisfy $(**)$ if, for all $k \in Z_+$,

(i) $\mu(kD) \geqslant \mu(D)$ for all measurable $D \subseteq G$,

(ii) $kC_N \subseteq C_{kN}$ for large N,

(iii) $\mu(C_N)/\mu(C_{kN}) > \delta_k$ for large N.

LEMMA 3. *Let G and $\mathscr{C}$ satisfy $(**)$. Given a measurable set $A \subseteq G$ of density 1, $k \in Z_+$, there is a measurable $A_k \subseteq G$ of density 1 such that*

$$A_k \cup kA_k \subseteq A.$$

Proof. Define

$$B_k = \{g \in A \mid kg \notin A\}.$$

Then

$$kB_k \subseteq A' \qquad \text{and} \qquad \mu(B_k \cap C_N) \leqslant \mu(kB_k \cap kC_N) \leqslant \mu(A' \cap C_{kN}).$$

Therefore

$$\frac{\mu(C_N)}{\mu(C_{kN})}\left(\frac{\mu(B_k \cap C_N)}{\mu(C_N)}\right) = \frac{\mu(B_k \cap C_N)}{\mu(C_{kN})}$$
$$\leqslant \frac{\mu(A' \cap C_{kN})}{\mu(C_{kN})} \to 0 \qquad \text{as} \qquad N \to \infty.$$

Since

$$\frac{\mu(C_N)}{\mu(C_{kN})} > \delta_k \qquad \text{for } N \text{ large},$$

we must have

$$\frac{\mu(B_k \cap C_N)}{\mu(C_N)} \to 0 \qquad \text{as} \qquad N \to \infty.$$

Choosing $A_k = B_k{}' \cap A$, the lemma is proved.

LEMMA 4. *Let G and $\mathscr{C}$ satisfy $(**)$ and $f \in \mathscr{B}$ with $M_{\mathscr{C}}(|f|) = 0$. Then for each $\delta > 0$ and $k \in Z_+$ there is $A_k \subseteq G$, $d(A_k) = 1$, such that*

$$|f(g)| < \delta \qquad \textit{and} \qquad |f(kg)| < \delta \quad \textit{for } g \in A_k.$$

Proof. Using Proposition 3, find B with $d(B) = 0$ such that

$$|f(g)| < \delta \qquad \text{for } g \notin (C_N \cup B).$$

Let

$$A = (C_N \cup B)' = C_N{}' \cap B'.$$

Since

$$\mu(C_N) < \infty, \qquad d(A) = 1.$$

Applying Lemma 3, there exists A_k with $d(A_k) = 1$ such that $A_k \cup kA_k \subseteq A$. Therefore

$$|f(g)| < \delta, \qquad |f(kg)| < \delta \qquad \text{for } g \in A.$$

Remark 3. Choose $G = R^m \times Z^n$ and $C_p \subseteq G$ with

$$C_p = \{(x_1, ..., x_{m+n}) \colon |x_i| < p, i = 1, ..., m+n\}, \qquad p \in Z_+.$$

Let σ_1 be Lebesgue measure on R^m and let σ_2 be counting measure on Z^n. Then $\mu = \sigma_1 \times \sigma_2$ is a Haar measure on G. We now show that $G, \mathscr{C} = (C_p)$, and μ satisfy (**). First we show $\mathscr{C}$ is rapidly expanding. Given

$$y = (y_1, \ldots, y_{m+n}) \qquad \text{and} \qquad p > 1/2[\max(|y_1|, \ldots, |y_{m+n}|) + 1],$$

$$(y + C_p) \cap C_p = \{(x_1, \ldots, x_{m+n}) \mid \max(y_i, 0) - p < x_i < p + \min(0, y_i)\},$$

$$\frac{\mu[(y + C_p) \cap C_p]}{\mu(C_p)}$$

$$= \frac{\prod_{i=1}^{m} (2p - |y_i|) \prod_{i=m+1}^{m+n} (2p - 1 - |y_i|)}{(2p)^m (2p-1)^n} \to 1 \qquad \text{as} \qquad p \to \infty.$$

Now to verify (**), for finite subsets of Z^n,

$$\sigma_2(kD) = \sigma_2(D) \qquad \text{for } k \in Z_+,$$

and for bounded subsets of R^m,

$$\sigma_1(kD) \geqslant \sigma_1(D).$$

Finally,

$$kC_p \subseteq C_{kp} \qquad \text{and} \qquad \frac{\mu(C_p)}{\mu(C_{kp})} = \frac{(2p)^m (2p-1)^n}{(2kp)^m (2p-1)^n} \to \frac{1}{k^m} \qquad \text{as} \qquad p \to \infty.$$

To obtain a simple representation of our groups, we now restrict attention to groups generated by a compact set, called compactly generated groups. By a well-known structure theorem [3, Theorem 9.8], we have

$$G \simeq R^m \times Z^n \times C, \qquad C \text{ compact}.$$

To further simplify, we would like to study $R^m \times Z^n$ with $\mathscr{C}$ as in Remark 3. In this case we can use all our previous results. What would happen to $\mathbf{H}_c^T$ if we "removed" the compact set C?

Remark 4. Let $G = A \times B$, G countable at infinity. Define unitary operators

$$U_a = T_{(a,e_2)}, \qquad V_b = T_{(e_1,b)},$$

where e_1, e_2 are the identity elements of A and B, respectively. We then have

$$T_{(a,b)} = U_a V_b = V_b U_a.$$

We wish to show that

$$\mathbf{H}_e^T = \mathbf{H}_e^U \cap \mathbf{H}_e^V.$$

Since any eigenvector of T is an eigenvector of both U and V, we have

$$\mathbf{H}_e^T \subseteq \mathbf{H}_e^U \cap \mathbf{H}_e^V.$$

Suppose $\mathbf{H}_e^T \neq \mathbf{H}_e^U \cap \mathbf{H}_e^V$. Choose f such that

$$f \in \mathbf{H}_e^U \cap \mathbf{H}_e^V \cap \mathbf{H}_c^T.$$

Since $\mathbf{H}_e^T \subseteq \mathbf{H}_e^U$, we may find a set $\{f_i\}$ of eigenvectors of U which spans $\mathbf{H}_e^U \cap \mathbf{H}_c^T$. We may then write

$$f = \sum_{i=1}^{\infty} a_i f_i = \sum_{i=1}^{\infty} b_i g_i$$

where g_i is an eigenvector of V.

We show that $\langle f_i, g_j\rangle = 0$. For, given $(a, b) \in A \times B$,

$$\begin{aligned} |\langle f_i, g_j\rangle| &= |\langle T_{(a,b)} f_i, T_{(a,b)} g_j\rangle| = |\langle V_b U_a f_i, U_a V_b g_j\rangle| \\ &= |\langle V_b f_i, U_a g_j\rangle| \\ &= |\langle T_{(-a,b)} f_i, g_j\rangle| . \end{aligned}$$

But, since $f_i \in \mathbf{H}_c^T$, by Proposition 2,

$$M_{\mathscr{G}}(|\langle T_g f_i, g_j\rangle|) = 0.$$

Therefore, we must have

$$|\langle f_i, g_j\rangle| = 0 \qquad \text{for all } i, j,$$

showing that $f = 0$.

Therefore,

$$\mathbf{H}_e^T = \mathbf{H}_e^U \cap \mathbf{H}_e^V, \qquad \mathbf{H}_c^T = \mathbf{H}_c^U \vee \mathbf{H}_c^V.$$

Now, suppose $A = R^m \times Z^n$, $B = C$, compact. Since B is compact, by Remark 2,

$$\mathbf{H}_c^V = \{0\}$$

giving

$$\mathbf{H}_c^T = \mathbf{H}_c^U.$$

4. Weakly Wandering Vectors and $\mathbf{H}_c^T$

Recall that $g_n \to \infty$ if, given any compact set K, there is $N \geqslant 0$ such that

$$g_n \notin K \qquad \text{for } n \geqslant N.$$

We define *weakly wandering vectors*, a concept central to our theorem, by the following.

DEFINITION 5. A vector $f \in \mathbf{H}$ is weakly wandering if there exists a sequence (g_n), $g_n \to \infty$, such that

$$\langle T_{g_m} f, T_{g_n} f \rangle = 0 \qquad \text{for } m \neq n.$$

We find that, under certain restrictions on the group G, these vectors are dense in $\mathbf{H}_c^T$. Let W^T denote the weakly wandering vectors.

THEOREM 1. *Let G be an Abelian, locally compact, compactly generated group and $\{T_g \mid g \in G\}$ a continuous group of unitary operators on $\mathbf{H}$. Then*

$$\mathbf{H}_c^T = \overline{W^T}.$$

Proof. We first show that $\overline{W^T} \subseteq \mathbf{H}_c^T$ by proving, for $f \in W^T$:

$$M_{\mathscr{C}}(|\langle T_g f, h \rangle|^2) = 0 \qquad \text{for all } h \in \mathbf{H} \tag{3.1}$$

and some $\mathscr{C}$ rapidly expanding.

We would then have, by Lemma 1,

$$M_{\mathscr{C}}(|\langle T_g f, h \rangle|) = 0 \qquad \text{for all } h \in \mathbf{H}.$$

By Proposition 2, it follows that

$$f \in \mathbf{H}_c^T.$$

Since $\mathbf{H}_c^T$ is a closed subspace

$$\overline{W^T} \subseteq \mathbf{H}_c^T.$$

We now prove property (3.1). Since G is countable at infinity, there exists some rapidly expanding sequence $\mathscr{C}$.

Assume $f \in \mathbf{H}$ is such that $\|f\| = 1$ and for some $(g_i) \in G$, $0 \leqslant i \leqslant r$, $g_0 = e$, we have

$$\langle T_{g_i} f, T_{g_j} f \rangle = 0, \qquad i \neq j, \quad i, j \leqslant r.$$

We will show that

$$\limsup_{n \to \infty} \frac{1}{\mu(C_n)} \int_{C_n} |\langle T_g f, h \rangle|^2 \, d\mu(g) \leqslant \frac{1}{r+1} \| h \|^2 \qquad \text{for } h \in \mathbf{H}.$$

Letting

$$f_i = T_{g_i} f$$

we find that

$$T_\ell f_0\,,\ T_\ell f_1\,, \ldots,\ T_\ell f_r$$

are orthogonal for all $\ell \in G$.

Therefore, by Bessel's inequality,

$$\sum_{i=0}^{r} |\langle T_\ell f_i\,, h\rangle|^2 \leqslant \| h \|^2.$$

For each $i \leqslant r$,

$$\int_{C_n} |\langle T_g f, h\rangle|^2\, d\mu(g) = \int_{C_n - g_i} |\langle T_g f_i\,, h\rangle|^2\, d\mu(g).$$

Summing over i, we have

$$(r+1) \int_{C_n} |\langle T_g f, h\rangle|^2\, d\mu(g) = \sum_{i=0}^{r} \int_{C_n - g_i} |\langle T_g f_i\,, h\rangle|^2\, d\mu(g).$$

These integrals add on their common range

$$D_{n,r} = \bigcap_{i=0}^{r} (C_n - g_i)$$

giving

$$\begin{aligned} &(r+1) \int_{C_n} |\langle T_g f, h\rangle|^2\, d\mu(g) \\ &\quad \leqslant \int_{D_{n,r}} \sum_{i=0}^{r} |\langle T_g f_i\,, h\rangle|^2\, d\mu(g) + \sum_{i=0}^{r} \int_{(C_n - g_i) \cap D'_{n,r}} |\langle T_g f_i\,, h\rangle|^2\, d\mu(g) \\ &\quad \leqslant \| h \|^2 \left(\mu(D_{n,r}) + \sum_{i=0}^{r} \mu[(C_n - g_i) \cap D'_{n,r}] \right). \end{aligned}$$

Let

$$D_{n,j} = \bigcap_{i=0}^{j} (C_n - g_i) \qquad \text{for } j \leqslant r.$$

Using induction on j, we show that

$$\lim_{n\to\infty} \frac{\mu[(C_n + g) \cap D'_{n,j}]}{\mu(C_n)} = 0, \qquad g \in G, \quad 0 \leqslant j \leqslant r. \tag{3.2}$$

Equation (3.2) clearly holds for $j = 0$, since $D_{n,0} = C_n$ and $\mathscr{C}$ is rapidly expanding.

Assume (3.2) is true for $j-1, j \leqslant r$. Then, since

$$D_{n,j} = D_{n,j-1} \cap (C_n - g_j)$$
$$D'_{n,j} = D'_{n,j-1} \cup (C_n{}' - g_j),$$

for $g \in G$,

$$\frac{\mu[(C_n + g) \cap D'_{n,j}]}{\mu(C_n)} \leqslant \frac{\mu[(C_n + g) \cap D'_{n,j-1}]}{\mu(C_n)} + \frac{\mu[(C_n + g + g_j) \cap C_n{}']}{\mu(C_n)} \to 0$$

as $n \to \infty$, proving (3.2).

Letting $g = e$, we therefore have

$$\frac{\mu(C_n \cap D_{n,r})}{\mu(C_n)} = \frac{\mu(D_{n,r})}{\mu(C_n)} \to 1$$

and finally

$$\limsup_{n\to\infty} \frac{1}{\mu(C_n)} \int_{C_n} |\langle T_g f, h\rangle|^2 \, d\mu(g) \leqslant \frac{1}{r+1} \| h \|^2 \qquad \text{for all } h \in \mathbf{H}. \tag{3.3}$$

Now, assume $f \in W^T$. Since for some sequence (g_n)

$$\langle T_{g_i} f, T_{g_j} f \rangle = 0, \qquad i \neq j,$$

we find (3.3) holds for all $r \geqslant 0$. Hence

$$M_{\mathscr{C}}(|\langle T_g f, h\rangle|^2) = 0 \qquad \text{for all } h \in \mathbf{H}.$$

Since (3.1) is true, we have proved

$$\overline{W^T} \subseteq \mathbf{H}_c{}^T.$$

Conversely, we show $\mathbf{H}_c{}^T \subseteq \overline{W^T}$. We may assume the group G is not compact, for by Remark 2, compact groups have

$$\mathbf{H}_c{}^T = \{0\}.$$

Since G is compactly generated, we may assume

$$G = R^m \times Z^n \times C.$$

By Remark 4, we may "remove" the compact set C and consider

$$G' = R^m \times Z^n,$$

and operators

$$T_g{}' = T_g \qquad \text{for } g \in G'$$

We then have

$$\mathbf{H}_c^{T'} = \mathbf{H}_c^{T}.$$

Since

$$W^{T'} \subseteq W^{T},$$

we have

$$\mathbf{H}_c^{T'} \subseteq \overline{W^{T'}} \Rightarrow \mathbf{H}_c^{T} \subseteq \overline{W^{T}}.$$

Therefore, without loss of generality we assume the group

$$G = R^m \times Z^n$$

and $\mathscr{C}$ is chosen as in Remark 3.

Given $f \in \mathbf{H}_c^T$ and $\epsilon > 0$, we construct a vector $f_\epsilon \in W^T$ such that

$$\| f - f_\epsilon \| < \epsilon.$$

The proof generalizes directly from the method for the case when $G = Z$, given by A. Ionescu Tulcea in an unpublished manuscript. It rests on the following Banach space theorem:

Let $\mathbf{B}$ be a Banach space, and denote for $\epsilon > 0$,

$$\mathbf{B}_\epsilon(h) = \{g \in \mathbf{B} \mid \| g - h \| < \epsilon\}.$$

PROPOSITION 4. *Given $f \in \mathbf{B}$ and mappings*

$$H_i: \mathbf{B} \to R_+ , \qquad \gamma_i: \mathbf{B}_\delta(f) \to \mathbf{B}, \qquad i = 1, 2$$

with constants δ, b_1, b_2 such that

$$\begin{aligned} &0 < \delta < 1 \\ &1 \leqslant b_1 \\ &0 < b_2 \leqslant \frac{1}{2(b_1^2 + 1)}, \end{aligned}$$

suppose that they satisfy:

(1) $H_1(f) = 0, \quad H_2(f) < \dfrac{\delta}{80 b_1^4};$

(2) $H_1(\gamma_1(h)) = 0$ *for all* $h \in \mathbf{B}_\delta(f)$;

(3) $\| h - \gamma_i(h) \| \leqslant b_1 H_i(h_2)$ *for all* $h \in \mathbf{B}_\delta(f)$, $i = 1, 2$;

(4) $| H_i(h_1) - H_i(h_2) | \leqslant b_1 \| h_1 - h_2 \|$ *for* $h_1, h_2 \in \mathbf{B}_\delta(f)$, $i = 1, 2$;

(5) $H_i(\gamma_2(h)) \leqslant \sum_{j=1}^{2} b_j H_j(h)$ *for all* $h \in \mathbf{B}_\delta(f)$, $i = 1, 2$.

Then there is $\delta' < \delta$ *such that, for* $h \in \mathbf{B}_{\delta'}(f)$, *we may define inductively*

$$h_0 = h, \qquad h_1 = \gamma_1(\gamma_2(h_0)),\ldots, h_{n+1} = \gamma_1(\gamma_2(h_n)),\ldots.$$

The sequence (h_n) *converges to a limit, denoted*

$$\varphi(h) = \lim_n h_n .$$

Letting

$$F = \max(H_1, H_2),$$

$$f' = \varphi(f), \qquad b' = 32b_1^4,$$

we find that $f' \in \mathbf{B}_{\delta'}(f)$ *and:*

(1′) $F(f') = 0$;

(2′) $F(\varphi(h)) = 0$ *for all* $h \in \mathbf{B}_{\delta'}(f)$;

(3′) $\| h - \varphi(h)\| \leqslant b'F(h)$ *for all* $h \in \mathbf{B}_{\delta'}(f)$;

(4′) $|F(h_1) - F(h_2)| \leqslant b_1 \| h_1 - h_2 \|$

$$\leqslant b' \| h_1 - h_2 \| \text{ for } h_1, h_2 \in \mathbf{B}_{\delta'}(f).$$

Proof. Let

$$\delta' = \frac{\delta}{80(b_1^2 + 1)^3}.$$

We show inductively that, for $h \in \mathbf{B}_{\delta'}(f)$, $n \geqslant 1$,

$$(\alpha_n) \begin{cases} h_0 = h \in \mathbf{B}_\delta(f), & \\ \gamma_2(h_0) \in \mathbf{B}_\delta(f), & h_1 = \gamma_1(\gamma_2(h_0)) \in \mathbf{B}_\delta(f), \\ \cdots & \\ \gamma_2(h_{n-1}) \in \mathbf{B}_\delta(f), & h_n = \gamma_1(\gamma_2(h_{n-1})) \in \mathbf{B}_\delta(f). \end{cases}$$

We first verify (α_1). It is evident that $h_0 \in \mathbf{B}_\delta(f)$.
To show that $\gamma_2(h_0) \in \mathbf{B}_\delta(f)$, we have

$$\begin{aligned} \| \gamma_2(h) - f \| &\leqslant \| \gamma_2(h) - h \| + \| h - f \| \\ &\leqslant b_1 H_2(h) + \| h - f \| \\ &\leqslant b_1[H_2(f) + b_1 \| h - f \|] + \| h - f \| \\ &\leqslant b_1 H_2(f) + (b_1^2 + 1)\,\delta' \\ &\leqslant b_1 \frac{\delta}{80 b_1^4} + \frac{(b_1^2+1)\,\delta}{80(b_1^2+1)^3} < \delta. \end{aligned}$$

Next, to show $h_1 \in \mathbf{B}_\delta(f)$,

$$\begin{aligned}
\| h_1 - f \| &= \| \gamma_1(\gamma_2(h)) - f \| \\
&\leqslant \| \gamma_1(\gamma_2(h)) - \gamma_2(h) \| + \| \gamma_2(h) - f \| \\
&\leqslant b_1 H_1(\gamma_2(h)) + \| \gamma_2(h) - f \| \\
&\leqslant b_1^2 \| \gamma_2(h) - f \| + \| \gamma_2(h) - f \| \\
&\leqslant (b_1^2 + 1)\, b_1 H_2(f) + (b_1^2 + 1)^2 \delta' \\
&\leqslant 2b_1^3 \frac{\delta}{80 b_1^4} + (b_1^2 + 1)^2 \frac{\delta}{80(b_1^2 + 1)^3} < \delta.
\end{aligned}$$

Now, suppose that (α_m) is verified. We show that (α_{m+1}) then holds. First, we need several inequalities:

(a) $F(\gamma_2(h)) \leqslant 2b_1 F(h)$. For,

$$H_i(\gamma_2(h)) \leqslant b_1 H_1(h) + b_2 H_2(h) \leqslant 2b_1 F(h).$$

(b) For $1 \leqslant n \leqslant m$, we have

$$H_1(h_n) = H_1(\gamma_1(\gamma_2(h_{n-1}))) = 0$$
$$F(\gamma_2(h_n)) \leqslant b_2 H_2(h_n), \qquad F(h_n) = H_2(h_n).$$

(c) $H_2(h_n) \leqslant (1/2)^{n-3}\, b_1^3 F(h)$ for $0 \leqslant n \leqslant m$. For the case $n = 1$, we find

$$\begin{aligned}
H_2(h_1) &= H_2(\gamma_1(\gamma_2(h_0))) \\
&\leqslant H_2(\gamma_2(h)) + b_1 \| \gamma_2(h) - \gamma_1(\gamma_2(h)) \| \\
&\leqslant H_2(\gamma_2(h)) + b_1^2 H_1(\gamma_2(h)) \\
&\leqslant (b_1^2 + 1)\, 2b_1 F(h) \leqslant 4b_1^3 F(h).
\end{aligned}$$

Now, assume we have shown (c) for the case $1 \leqslant n \leqslant m - 1$. We show it for $n + 1$. Remembering that

$$b_2(1 + b_1^2) \leqslant \tfrac{1}{2}, \qquad H_1(h_n) = 0,$$

we have

$$\begin{aligned}
H_2(h_{n+1}) &= H_2(\gamma_1(\gamma_2(h_n))) \\
&\leqslant H_2(\gamma_2(h_n)) + b_1 \| \gamma_2(h_n) - \gamma_1(\gamma_2(h_n)) \| \\
&\leqslant H_2(\gamma_2(h_n)) + b_1^2 H_1(\gamma_2(h_n)) \\
&\leqslant (1 + b_1^2)\, b_2 H_2(h_n) \\
&\leqslant \tfrac{1}{2} H_2(h_n) \leqslant (\tfrac{1}{2})^{n-2}\, b_1^3 F(h).
\end{aligned}$$

From (b) and (c), we deduce:

(d) $F(h_n) \leqslant (\frac{1}{2})^{n-3} b_1^3 F(h)$ for $0 \leqslant n \leqslant m$.

(e) For $0 \leqslant k < n \leqslant m + 1$, assuming

$$\gamma_2(h_m) \in \mathbf{B}_\delta(f),$$

we have

$$\| h_n - h_k \| \leqslant \left[\sum_{j=k}^{n-1} (\tfrac{1}{2})^{j-4} \right] b_1^4 F(h) \leqslant 32 b_1^4 F(h).$$

We have

$$\begin{aligned} \| h_1 - h_0 \| &\leqslant \| \gamma_1(\gamma_2(h)) - \gamma_2(h) \| + \| \gamma_2(h) - h \| \\ &\leqslant b_1[H_1(\gamma_2(h)) + H_2(h)] \\ &\leqslant b_1[2b_1F(h) + F(h)] \\ &\leqslant (\tfrac{1}{2})^{-4} b_1^4 F(h). \end{aligned}$$

If $j \geqslant 1$, we have by (b) and (d)

$$\begin{aligned} \| h_{j+1} - h_j \| &\leqslant \| \gamma_1(\gamma_2(h_j)) - \gamma_2(h_j) \| + \| \gamma_2(h_j) - h_j \| \\ &\leqslant b_1[F(\gamma_2(h_j)) + F(h_j)] \\ &\leqslant b_1[b_2F(h_j) + F(h_j)] \\ &\leqslant 2b_1F(h_j) \leqslant (\tfrac{1}{2})^{j-4} b_1^4 F(h). \end{aligned}$$

Therefore, we have

$$\| h_n - h_k \| \leqslant \left[\sum_{j=k}^{n-1} (\tfrac{1}{2})^{j-4} \right] b_1^4 F(h).$$

Now, to verify (α_{m+1}).

$$\begin{aligned} \| \gamma_2(h_m) - f \| &\leqslant \| \gamma_2(h_m) - h_m \| + \| h_m - h \| + \| h - f \| \\ &\leqslant b_1 H_2(h_m) + 32 b_1^4 F(h) + \delta' \\ &\leqslant 40 b_1^4 F(h) + \delta'. \end{aligned}$$

Since

$$F(h) \leqslant H_2(f) + b_1\delta', \qquad H_2(f) < \frac{\delta}{80 b_1^4},$$

we then have

$$\begin{aligned} \| \gamma_2(h_m) - f \| &\leqslant 40 b_1^4 [H_2(f) + b_1\delta'] + \delta' \\ &\leqslant 40 b_1^4 \frac{\delta}{80 b_1^4} + (40 b_1^5 + 1)\,\delta' \\ &\leqslant \frac{\delta}{2} + 40(b_1^2 + 1)^3 \frac{\delta}{80(b_1^2 + 1)^3} \\ &< \delta. \end{aligned}$$

Next, by (e) we have

$$\begin{aligned} \| h_{m+1} - f \| &\leqslant \| h_{m+1} - h \| + \| h - f \| \\ &\leqslant 32 b_1^4 F(h) + \delta' < \delta \end{aligned}$$

by the above. Therefore, (α_{m+1}) holds.

Inequality (e) shows us further that (h_m) is Cauchy, converging to a limit

$$\varphi(h) = \lim_m h_m .$$

By (4), H_1 and H_2 are continuous; using also (d), we find

$$\begin{aligned} F(\varphi(h)) &= \lim_m F(h_m) \\ &\leqslant \lim_m (\tfrac{1}{2})^{m-3}\, b_1^3 F(h) \\ &= 0 \qquad \text{for } h \in \mathbf{B}_{\delta'}(f). \end{aligned}$$

Letting $b' = 32 b_1^4$, we easily verify:

(1′) $F(f') = F(\varphi(f)) = 0$;

(2′) $F(\varphi(h)) = 0$ for $h \in \mathbf{B}_{\delta'}(f)$;

(3′) $\| h - \varphi(h) \| = \lim_m \| h - h_m \|$

$$\leqslant 32 b_1^4 F(h) = b' F(h) \quad \text{for} \quad h \in \mathbf{B}_{\delta'}(f);$$

(4′) $| F(h_1) - F(h_2) | \leqslant b_1 \| h_1 - h_2 \|$ for $h_1, h_2 \in \mathbf{B}_{\delta'}(f)$.

The proof of Proposition 4 is now complete.

Remark 5. Let $f \in \mathbf{H}$, $\| f \| < 2$, $0 < \delta < 1$. If (S_i) and (U_j) are sequences of linear contractions on $\mathbf{H}$, then for $h_1, h_2 \in \mathbf{B}_\delta(f)$ and $i \neq j$,

$$|\langle S_i h_1 , U_j h_1 \rangle| \leqslant 6 \| h_1 - h_2 \| + |\langle S_i h_2 , U_j h_2 \rangle| .$$

Therefore, for $h_1, h_2 \in \mathbf{B}_\delta(f)$,

$$\Big| \max_{\substack{i,j \leqslant n \\ i \neq j}} |\langle S_i h_1 , U_j h_1 \rangle| - \max_{\substack{i,j \leqslant n \\ i \neq j}} |\langle S_i h_2 , U_j h_2 \rangle| \Big| \leqslant 6 \| h_1 - h_2 \| .$$

Proof.

$$\langle S_i h_1 , U_j h_1 \rangle = \langle S_i (h_1 - h_2), U_j h_1 \rangle + \langle S_i h_2 , U_j (h_1 - h_2) \rangle + \langle S_i h_2 , U_j h_2 \rangle.$$

Therefore,

$$\begin{aligned} |\langle S_i h_1 , U_j h_1 \rangle| &\leqslant \| h_1 - h_2 \| (\| h_1 \| + \| h_2 \|) + |\langle S_i h_2 , U_j h_2 \rangle| \\ &\leqslant 6 \| h_1 - h_2 \| + |\langle S_i h_2 , U_j h_2 \rangle| . \end{aligned}$$

Returning to the proof of our theorem, given $f \in \mathbf{H}_c^T$ such that $\|f\| = 1$, for any ϵ, $0 < \epsilon < 1$, we inductively construct sequences (f_n), (g_n) such that

$$f_n \in \mathbf{H}, \quad \|f_n - f_{n-1}\| < \epsilon/2^n; \qquad g_n \in G, \quad g_n \to \infty$$
$$\langle T_{g_i} f_n , T_{g_j} f_n \rangle = 0 \qquad \text{for } i, j \leqslant n, \quad i \neq j.$$

Finally, defining

$$f_\epsilon = \lim_{n \to \infty} f_n$$

we find that

$$\|f - f_\epsilon\| < \epsilon, \qquad \langle T_{g_i} f_\epsilon , T_{g_j} f_\epsilon \rangle = 0 \quad \text{for } i \neq j.$$

Therefore

$$f_\epsilon \in W^T.$$

Specifically, letting $\mathbf{B} = \mathbf{H}_c^T$, we may inductively construct sequences (f_m), (g_m), (ϵ_m), (φ_m) such that, for all m,

$$g_m \in G - C_m , \qquad g_0 = e, \qquad C_0 = \varnothing$$
$$\epsilon_m < \epsilon_{m-1}/2 < \epsilon/2^{m+1}, \qquad \epsilon_0 < \epsilon/2$$
$$f_m \in \mathbf{B}, \qquad \|f_m - f_{m-1}\| < \epsilon_{m-1} , \qquad f_0 = f$$
$$\varphi_m \colon \mathbf{B}_{\epsilon_m}(f_m) \to \mathbf{B}.$$

Further, defining

$$F_m(h) = \max_{\substack{i,j \leqslant m \\ i \neq j}} |\langle T_{g_i} h, T_{g_j} h \rangle|$$
$$F_0 \equiv 0$$

we have

(a) $F_m(f_m) = 0$

(b) $F_m(\varphi_m(h)) = 0$ for all $h \in \mathbf{B}_{\epsilon_m}(f_m)$

(c) $\| h - \varphi_m(h)\| \leqslant D_m F_m(h)$ for $D_m > 0$ and for all $h \in \mathbf{B}_{\epsilon_m}(f_m)$. $\quad (\mathrm{I_m})$

(I_0) is satisfied by choosing $\varphi_0(h) = h$. Therefore, assume sequences have been constructed through n, satisfying (I_m) for $m \leqslant n$. Define

$$H \colon G \times \mathbf{H} \to R_+$$
$$\gamma \colon G \times \mathbf{H} \to \mathbf{H}$$

by

$$H(g, h) = \max_{i \leqslant n} |\langle T_{g_i} h, T_g h \rangle|$$

$$\gamma(g, h) = h + \sum_{j=0}^{n} a_j(g, h)\, T_{g-g_j} h$$

where

$$a_j(g, h) = -\frac{\langle T_{g_j}h, T_g h\rangle}{\| h \|^2}.$$

We show that, for an appropriate g_{n+1}, we may define

$$\begin{aligned} H_2(\cdot) &= H(g_{n+1}, \cdot) \\ H_1 &= F_n \\ \gamma_2(\cdot) &= \gamma(g_{n+1}, \cdot) \\ \gamma_1 &= \varphi_n \end{aligned}$$

to satisfy the assumptions of Proposition 4. We first calculate bounds on some quantities. For $h \in \mathbf{B}_{\epsilon_n}(f_n)$,

$$\| f_n - f \| < \epsilon/2 - \epsilon/2^{n+1}$$
$$\| h - f \| < \epsilon/2, \qquad \tfrac{1}{2} \leqslant \| h \| \leqslant 2.$$

Letting $a_j = a_j(g, h)$ for simplicity,

$$| a_j | = \frac{|\langle T_{g_j}h, T_g h\rangle|}{\| h \|^2} \leqslant 1$$

$$| a_j |^2 \leqslant | a_j | \leqslant \frac{H(g, h)}{\| h \|^2} < 4H(g, h)$$

$$| a_j \bar{a}_k | \leqslant \frac{H(g, h)^2}{\| h \|^4} \leqslant \frac{4H(g, h)^2}{\| h \|^2}.$$

Let us further define

$$H^*(g, h) = \max_{k, \ell, i \leqslant n} \{|\langle T_{g_k + g_\ell - g_i}h, T_g h\rangle|, |\langle T_{g_k + g_\ell}h, T_{2g}h\rangle|\}.$$

To find bounds on $F_n(\gamma(g, h))$ and $H(g, \gamma(g, h))$ we first calculate, for $i \leqslant n$, $g' \in G$,

$$\begin{aligned} &\langle T_{g_i}\gamma(g, h), T_{g'}\gamma(g, h)\rangle \\ &= \left\langle T_{g_i}\left(h + \sum_{\ell=0}^{n} a_\ell T_{g-g_\ell}h\right), T_{g'}\left(h + \sum_{j=0}^{n} a_j T_{g-g_j}h\right)\right\rangle \\ &= \langle T_{g_i}h, T_{g'}h\rangle + a_i\langle T_{g_i}(T_{g-g_i}h), T_{g'}h\rangle \\ &\quad + \sum_{\ell \neq i} a_\ell\langle T_{g_i}(T_{g-g_\ell}h), T_{g'}h\rangle + \sum_{j=0}^{n} \bar{a}_j\langle T_{g_i}h, T_{g'}(T_{g-g_j}h)\rangle \\ &\quad + \sum_{j,\ell \leqslant n} a_\ell \bar{a}_j\langle T_{g_i}(T_{g-g_\ell}h), T_{g'}(T_{g-g_j}h)\rangle. \end{aligned}$$

First taking $g' = g_k$ for $k \neq i$, $k \leqslant n$, since $H(g, h) \leqslant H^*(g, h)$, we find:

$$|\langle T_{g_i}h, T_{g_k}h\rangle| \leqslant F_n(h);$$

$$\begin{aligned}
&| a_i | \, |\langle T_{g_i}(T_{g-g_i}h), T_{g_k}h\rangle| \\
&\qquad \leqslant 4H(g, h) \, |\langle T_g h, T_{g_k}h\rangle| \leqslant 4H(g, h) \, H^*(g, h); \\
&\sum_{\ell \neq i} | a_\ell | \, |\langle T_{g+g_i-g_\ell}h, T_{g_k}h\rangle| \\
&\qquad \leqslant 4H(g, h) \sum_{\ell \neq i} |\langle T_g h, T_{g_k+g_\ell-g_i}h\rangle| \leqslant 4nH(g, h) \, H^*(g, h); \\
&\sum_{j \leqslant n} | a_j | \, |\langle T_{g_i}h, T_{g_k}(T_{g-g_j}h)\rangle| \\
&\qquad \leqslant 4H(g, h) \sum_{j \leqslant n} |\langle T_{g_i+g_j-g_k}h, T_g h\rangle| \leqslant 4(n + 1) \, H(g, h) \, H^*(g, h); \\
&\sum_{j,\ell \leqslant n} | a_\ell \bar{a}_j | \, |\langle T_{g_i}(T_{g-g_\ell}h), T_{g_k}(T_{g-g_j}h)\rangle| \\
&\qquad \leqslant \frac{4H(g, h)^2}{\| h \|^2} \sum_{j,\ell \leqslant n} |\langle T_{g_i+g_j-g_k-g_\ell}h, h\rangle| \leqslant 4(n + 1)^2 \frac{H(g, h)^2}{\| h \|^2} \| h \|^2 \\
&\qquad \leqslant 4(n + 1)^2 \, H(g, h) \, H^*(g, h).
\end{aligned}$$

By Remark 5,

$$H^*(g, h) \leqslant 6 \| f_n - h \| + H^*(g, f_n).$$

We therefore have

$$\begin{aligned}
F_n(\gamma(g, h)) &\leqslant F_n(h) + 8(n + 1) \, H^*(g, h) \, H(g, h) + 4(n + 1)^2 \, H^*(g, h) \, H(g, h) \\
&\leqslant F_n(h) + 4(n + 1) \, (n + 3) \, [6 \| h - f_n \| + H^*(g, f_n)] \, H(g, h)
\end{aligned} \tag{3.4}$$

Second, taking instead $g' = g$, we have:

$$\begin{aligned}
&\langle T_{g_i}h, T_g h\rangle + a_i \langle T_{g_i}(T_{g-g_i}h), T_g h\rangle \\
&\qquad = \langle T_{g_i}h, T_g h\rangle - \frac{\langle T_{g_i}h, T_g h\rangle}{\| h \|^2} \| h \|^2 = 0; \\
&\left| \sum_{\ell \neq i} a_\ell \langle T_{g_i}(T_{g-g_\ell}h), T_g h\rangle \right| \\
&\qquad \leqslant \sum_{\ell \neq i} | a_\ell | \, |\langle T_{g_i}h, T_{g_\ell}h\rangle| \leqslant nF_n(h); \\
&\sum_{j \leqslant n} | a_j | \, |\langle T_{g_i}h, T_g(T_{g-g_j}h)\rangle| \\
&\qquad \leqslant 4H(g, h) \sum_{j \leqslant n} |\langle T_{g_i+g_j}h, T_{2g}h\rangle| \leqslant 4(n + 1) \, H(g, h) \, H^*(g, h); \\
&\sum_{j,\ell \leqslant n} | a_\ell a_j | \, |\langle T_{g_i}(T_{g-g_\ell}h), T_g(T_{g-g_j}h)\rangle| \\
&\qquad \leqslant \frac{4H(g, h)^2}{\| h \|^2} \sum_{j,\ell \leqslant n} |\langle T_{g_i+g_j-g_\ell}h, T_g h\rangle| \leqslant \frac{4(n + 1)^2 \, H(g, h)^2}{\| h \|^2} \| h \|^2 \\
&\qquad \leqslant 4(n + 1)^2 \, H^*(g, h) \, H(g, h).
\end{aligned}$$

Combining these inequalities, we have

$$\begin{aligned}
&H(g, \gamma(g, h)) \\
&\quad \leqslant nF_n(h) + 4(n+1)\, H^*(g, h)\, H(g, h) + 4(n+1)^2\, H^*(g, h)\, H(g, h) \\
&\quad \leqslant nF_n(h) + 4(n+1)(n+2)\, [6 \| h - f_n \| + H^*(g, f_n)]\, H(g, h). \qquad (3.5)
\end{aligned}$$

We now choose functions and constants to satisfy the hypotheses of Proposition 4:

$$\mathbf{B} = \mathbf{H}_c^T; \qquad f = f_n; \qquad b_1 > \max(D_n, 4(n+1), 6);$$

$$c_1 < \frac{1}{16(b^2+1)(n+1)(n+3)};$$

$$\delta < \min\left(\frac{c_1}{6}, \frac{\epsilon_n}{2}\right); \qquad c_2 = \frac{\delta}{80 b_1^4}; \qquad c = \min(c_1, c_2);$$

$$b_2 = 8(n+1)(n+3)\, c_1 < \frac{1}{2(b_1^2+1)}.$$

Remark 6. To satisfy the hypotheses, we must be able to find $g \in G$ such that

$$H^*(g, f_n) < c.$$

Since

$$f_{k,\ell,i}(g) = \langle T_{g_k + g_\ell - g_i} f_n, T_g f_n \rangle$$

satisfies

$$M_{\mathscr{G}}(|f_{k,\ell,i}|) = 0,$$

we have, by Lemma 4, a set $A_{k,\ell,i}$ of density 1 such that

$$|f_{k,\ell,i}(g)| < c$$

and

$$|f_{k,\ell,i}(2g)| < c \qquad \text{for } g \in A_{k,\ell,i}.$$

By Properties (i) and (iii) of density zero sets,

$$A = C'_{n+1} \cap \left(\bigcap_{k,\ell,i \leqslant n} A_{k,\ell,i} \right)$$

has density 1. Thus, choosing any $g_{n+1} \in A$, we have

$$\begin{aligned}
H^*(g_{n+1}, f_n) &= \max_{k,\ell,i \leqslant n} \{ |\langle T_{g_k + g_\ell - g_i} f_n, T_{g_{n+1}} f_n \rangle|, |\langle T_{g_k + g_\ell} f_n, T_{2g_{n+1}} f_n \rangle| \} \\
&\leqslant \max_{k,\ell,i \leqslant n} \{ |f_{k,\ell,i}(g_{n+1})|, |f_{k,\ell,i}(2g_{n+1})| \} < c.
\end{aligned}$$

Finally, let

$$H_1 = F_n;$$
$$H_2(\cdot) = H(g_{n+1}, \cdot);$$
$$\gamma_1 = \varphi_n;$$
$$\gamma_2(\cdot) = \gamma(g_{n+1}, \cdot).$$

We verify each hypothesis

(1) $H_1(f) = 0, \quad H_2(f) < \delta/80b_1^4.$

By $I_n(a)$,

$$F_n(f_n) = H_1(f) = 0.$$

By Remark 6,

$$H(g_{n+1}, f_n) \leqslant H^*(g_{n+1}, f_n) < c < \frac{\delta}{80b_1^4}.$$

(2) $H_1(\gamma_1(h)) = 0$ for all $h \in \mathbf{B}_\delta(f)$

By $I_n(b)$,

$$F_n(\varphi_n(h)) = 0 \qquad \text{for all } h \in \mathbf{B}_{\epsilon_n}(f_n),$$

and

$$\mathbf{B}_\delta(f) \subseteq \mathbf{B}_{\epsilon_n}(f_n).$$

(3) $\| h - \gamma_i(h) \| \leqslant b_1 H_i(h)$ for all $h \in \mathbf{B}_\delta(f)$, $i = 1, 2$.

For, by $I_n(c)$,

$$\| h - \varphi_n(h) \| \leqslant D_n F_n(h) \leqslant b_1 F_n(h) \qquad \text{for } h \in \mathbf{B}_{\epsilon_n}(f_n).$$

Since

$$| a_j(g_{n+1}, h) | \leqslant 4H(g_{n+1}, h),$$

we have

$$\| h - \gamma_2(h) \| \leqslant \sum_{j=0}^{n} | a_j(g_{n+1}, h) |$$
$$\leqslant 4(n+1)\, H(g_{n+1}, h) \leqslant b_1 H_2(h).$$

(4) $| H_i(h_1) - H_i(h_2) | \leqslant b_1 \| h_1 - h_2 \|$ for $h_1, h_2 \in \mathbf{B}_\delta(f)$, $i = 1, 2$.
First, let

$$S_i = T_{g_i} = U_i .$$

By Remark 5, since

$$H_1(h) = \max_{\substack{i,j \leqslant n \\ i \neq j}} |\langle S_i h, U_j h \rangle| ,$$

we have

$$| H_1(h_1) - H_1(h_2)| \leqslant 6 \| h_1 - h_2 \| < b_1 \| h_1 - h_2 \| .$$

Next, let

$$S_i = T_{g_i}, \qquad U_j = T_{g_{n+1}} .$$

We have

$$\begin{aligned} H_2(h) &= \max_{i \leqslant n} |\langle T_{g_i} h, T_{g_{n+1}} h\rangle| \\ &= \max_{\substack{i,j \leqslant n \\ i \neq j}} |\langle S_i h, U_j h\rangle| , \end{aligned}$$

and therefore

$$| H_2(h_1) - H_2(h_2)| \leqslant 6 \| h_1 - h_2 \| < b_1 \| h_1 - h_2 \| .$$

$$(5) \quad H_i(\gamma_2(h)) \leqslant \sum_{i=1}^{2} b_i H_i(h) \text{ for } h \in \mathbf{B}_\delta(f),\ i = 1, 2.$$

Since

$$\delta < c_1/6$$

and, by Remark 6,

$$H^*(g_{n+1}, f_n) < c_1 ,$$

we have

$$4(n+1)(n+3)[6\delta + H^*(g_{n+1}, f_n)] < 8(n+1)(n+3)c_1 = b_2 .$$

Therefore, by (3.4) we have

$$\begin{aligned} H_1(\gamma_2(h)) &= F_n(\gamma(g_{n+1}, h)) \\ &\leqslant F_n(h) + b_2 H(g_{n+1}, h) \\ &\leqslant b_1 H_1(h) + b_2 H_2(h). \end{aligned}$$

By (3.5), we have

$$\begin{aligned} H_2(\gamma_2(h)) &= H(g_{n+1}, \gamma(g_{n+1}, h)) \\ &\leqslant n F_n(h) + b_2 H(g_{n+1}, h) \\ &< b_1 H_1(h) + b_2 H_2(h). \end{aligned}$$

Therefore, by Proposition 4, there exist

$$\begin{gathered} \delta' < \delta, \qquad f' \in \mathbf{B}_{\delta'}(f), \\ F(h) = \max_{\substack{i,j \leqslant n+1 \\ i \neq j}} |\langle T_{g_i} h, T_{g_j} h\rangle| , \\ \varphi\colon \mathbf{B}_{\delta'}(f) \to \mathbf{B}, \qquad b' \geqslant b_1 , \end{gathered}$$

satisfying (1′)–(4′).

Let

$$f_{n+1} = f'$$

and choose ϵ_{n+1} so that

$$\epsilon_{n+1} < \delta' < \epsilon_n/2 \qquad \text{and} \qquad \mathbf{B}_{\epsilon_{n+1}}(f_{n+1}) \subseteq \mathbf{B}_{\delta'}(f_n).$$

Letting

$$D_{n+1} = b', \qquad \varphi_{n+1} = \varphi, \qquad F_{n+1} = F,$$

we find from (1′)–(4′) that

$$I_{n+1}(a)\text{–}(c) \text{ hold}$$

and therefore (I_{n+1}) is true.

We then have a sequence (f_n) with

$$f_n \to f_\epsilon, \qquad \|f_\epsilon - f\| < \epsilon, \qquad f_\epsilon \in W^T.$$

Therefore

$$f \in \overline{W^T}$$

and finally

$$\mathbf{H}_c{}^T \subseteq \overline{W^T}.$$

The proof of Theorem 1 is now complete.

5. Examples

The following examples illustrate Theorem 1 and suggest a possible way to strengthen the theorem.

Example 1. Consider the space $L^2(R)$ of square-integrable functions over the reals, with Lebesgue measure. Define a group of unitary operators $\{T_\alpha \mid \alpha \in R\}$ on $L^2(R)$ by

$$T_\alpha f(x) = f(x + \alpha) \qquad \text{for } f \in L^2(R), \quad \alpha, x \in R.$$

$\{T_\alpha \mid \alpha \in R\}$ has no eigenvectors. For, suppose

$$f(x + \alpha) = c^\alpha f(x), \qquad \alpha \in R, \quad |c| = 1.$$

Then

$$\begin{aligned} \int |f(x)|^2 \, dx &= \sum_{n=-\infty}^{\infty} \int_n^{n+1} |f(x)|^2 \, dx \\ &= \sum_{n=-\infty}^{\infty} \int_n^{n+1} |f(x - n)|^2 \, dx \\ &= \sum_{n=-\infty}^{\infty} \int_0^1 |f(x)|^2 \, dx = 0 \quad \text{or} \quad \infty. \end{aligned}$$

Since $\int |f(x)|^2 \, dx < \infty$, we must have

$$f = 0 \text{ a.e.}$$

Therefore, the continuous part is all of $L^2(R)$.

Now, we show that functions with compact support are weakly wandering. Suppose $g \in L^2(R)$ is such that

$$g(x) = 0 \qquad \text{for } |x| > N.$$

Then we have, for $|\alpha_1 - \alpha_2| > 2N$, $\alpha_1, \alpha_2 \in R$,

$$\langle T_{\alpha_1} g, T_{\alpha_2} g \rangle = \int_{-N}^{N} g(x + \alpha_1 - \alpha_2) \, g(x) \, dx = 0.$$

Therefore, if (α_n) is any increasing sequence in R with $(\alpha_{n+1} - \alpha_n) > 2N$, we have

$$T_{\alpha_i} g \perp T_{\alpha_j} g \qquad \text{for } i \neq j.$$

Thus functions with compact support are weakly wandering vectors dense in $L^2(R)$.

Example 2. We consider the space $L^2(\mathbf{T})$ of square-integrable functions on the circle group $\mathbf{T}$. $L^2(\mathbf{T})$ can be identified with the functions in $L^2(R)$ with period 2π. For $f \in L^2(\mathbf{T})$, $\alpha \in R$, we define $T_\alpha : f \to T_\alpha f$ as:

$$T_\alpha f(x) = e^{i\alpha x} f(x) \qquad \text{for } x \in [0, 2\pi).$$

$\{T_\alpha \mid \alpha \in R\}$ has no eigenvectors. For, assume f is such that

$$T_\alpha f(x) = e^{i\alpha\theta} f(x) \qquad \text{for some } \theta \in R.$$

We then have

$$e^{i\alpha\theta} f(x) = e^{i\alpha x} f(x)$$

so that

$$f = 0 \text{ a.e.}$$

We now show that, for any (a_n), a_n complex, the function

$$f_M(x) = \sum_{n=-M}^{M} a_n e^{inx} \qquad x \in [0, 2\pi)$$

is weakly wandering with respect to $\{T_\alpha \mid \alpha \in R\}$.

Given k_1, k_2 integers such that $| k_1 - k_2 | > 2M$,

$$\begin{aligned} \langle T_{k_1} f_M, T_{k_2} f_M \rangle &= \sum_{n=-M}^{M} \sum_{m=-M}^{M} a_n \bar{a}_m \langle e^{i(k_1+n)x}, e^{i(k_2+m)x} \rangle \\ &= \sum_{n=-M}^{M} \sum_{m=-M}^{M} a_n \bar{a}_m \int_0^{2\pi} e^{i(k_1-k_2+n-m)x} \, dx \\ &= 0 \end{aligned}$$

since $k_1 - k_2 + n - m$ is a nonzero integer. The trigonometric polynomials are dense in $L^2(\mathbf{T})$, again demonstrating our theorem.

EXAMPLE 3. We consider the same transformation as in Example 2, defined on the larger space $L^2(R)$. Again, there can be no eigenvectors.

To construct a dense set of weakly wandering vectors, we again draw on tools from harmonic analysis [5, pp. 126, 142]. Let $\hat{f}$ denote the Fourier transform of f, and $f * g$ the convolution of f and g. Recall that, for $f \in L^1(R)$, $g \in L^p(R)$, $(1 \leqslant p < \infty)$,

$$f * g \in L^p(R) \qquad \text{and} \qquad \widehat{f * g} = \hat{f}\hat{g}.$$

Let K_λ be the Féjer kernel, where

$$\begin{aligned} K(x) &= \frac{1}{2\pi} \left(\frac{\sin x/2}{x/2} \right), \qquad && x \in R \\ K_\lambda(x) &= \lambda K(\lambda x), && \lambda \in R_+ . \end{aligned}$$

Then $K_\lambda \in L^1$, and for $f \in L^2(R)$,

$$K_\lambda * f \to f \quad \text{in} \quad \| \, \|_2 \qquad \text{as} \qquad \lambda \to \infty.$$

In addition,

$$\hat{K}_\lambda(\xi) = 0 \qquad \text{for } | \xi | > \lambda.$$

Therefore, for $f \in L^2(R)$,

$$\widehat{K_\lambda * f}(\xi) = \hat{K}_\lambda(\xi) \hat{f}(\xi) = 0 \qquad \text{for } | \xi | > \lambda.$$

Now, given any $f \in L^2(R)$, we may choose $f' \in L^1 \cap L^2$ such that

$$\| f - f' \|_2 < \epsilon/2.$$

Then

$$K_\lambda * f' \in L^1 \cap L^2 \qquad \text{for all } \lambda,$$

and

$$K_\lambda * f' \to f' \quad \text{in} \quad \| \, \|_2 \qquad \text{as} \qquad \lambda \to \infty.$$

Choose λ so that

$$\| f' - K_\lambda * f' \|_2 < \epsilon/2$$

and set

$$f_\epsilon = K_\lambda * f' \in L^1 \cap L^2.$$

f_ϵ is then weakly wandering. To show this, recall that $\hat{f}_\epsilon$ vanishes off of $[-\lambda, \lambda]$. Set

$$g_\alpha(x) = e^{-i\alpha x} f_\epsilon(x) \qquad \text{for } \alpha \in R.$$

Then

$$\hat{g}_\alpha(\xi) = \hat{f}_\epsilon(\xi + \alpha).$$

Using Parseval's formula, for $|\alpha| > 2\lambda$ we have

$$\begin{aligned}
\langle T_\alpha f_\epsilon , f_\epsilon \rangle &= \int T_\alpha f_\epsilon(x) \overline{f_\epsilon(x)} \, dx \\
&= \int f_\epsilon(x) \, \overline{e^{-i\alpha x} f_\epsilon(x)} \, dx \\
&= \frac{1}{2\pi} \int \hat{f}_\epsilon(\xi) \, \overline{\hat{g}_\alpha(\xi)} \, d\xi \\
&= \frac{1}{2\pi} \int_{-\lambda}^{\lambda} \hat{f}_\epsilon(\xi) \overline{\hat{f}_\epsilon(\xi + \alpha)} \, d\xi = 0.
\end{aligned}$$

Choosing any increasing sequence (α_n) with $(\alpha_{n+1} - \alpha_n) > 2\lambda$,

$$T_{\alpha_i} f_\epsilon \perp T_{\alpha_j} f_\epsilon \qquad \text{for } i \neq j.$$

Therefore f_ϵ is weakly wandering and

$$\| f - f_\epsilon \| < \epsilon.$$

Examples 1 and 3 suggest a new definition of weakly wandering that could yield a stronger version of Theorem 1.

Define a vector f as weakly wandering if there exists a sequence (D_n) of disjoint open sets in G such that, for any compact set $K \subseteq G$,

$$D_n \subseteq G - K \qquad \text{for } n \text{ large}$$

and

$$T_{g_i} f \perp T_{g_j} f \qquad \text{for } g_i \in D_i , \quad g_j \in D_j , \quad i \neq j.$$

In examples 1 and 3, each weakly wandering vector we found satisfies this stronger requirement. In fact, we showed for each such f, that there exists $N > 0$ such that

$$\langle T_{\alpha_1} f, T_{\alpha_2} f \rangle = 0 \qquad \text{for } | \alpha_1 - \alpha_2 | > N.$$

Thus, any sequence of open sets separated by intervals of length N would show f is weakly wandering in this new sense. The proof of Theorem 1 using this stronger definition remains to be explored.

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References

1. W. Ambrose, Spectral resolution of groups of unitary operators, *Duke Math. J.* **11** (1944), 589–595.
2. P. Halmos, "Introduction to Hilbert Space," Chelsea, New York, 1951.
3. E. Hewitt and K. A. Ross, "Abstract Harmonic Analysis I," *Grund, der Math. Wiss.* **15**, Springer, Verlag, Berlin, (1963).
4. E. Hewitt and K. R. Stromberg, A remark on Fourier–Stieltjes transforms, *Anais, Acad. Brasil. Ci.* **34** (1962), 175–180.
5. Y. Katznelson, "An Introduction to Harmonic Analysis," Wiley, New York, (1968).
6. U. Krengel, Weakly wandering vectors and weakly independent partitions, *Amer. Math. Soc. Trans.* **164** (1972), 199–226.